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Note

On Distortion Functions for the Strong Constraint Method of Numerically Generating Orthogonal Coordinate Grids

1. Introduction

An important advance in the application of finite difference techniques to complicated problems in fluid mechanics has been the development of methods to construct boundary-fitted curvilinear coordinates so that solutions can be obtained on domains of quite general shape without loss of accuracy in the application of boundary conditions. Extensive surveys of research in grid generation are presented in Thompson *et al.* [1] and Eiseman [2], where the latter focuses on the importance of grid generation techniques in fluid mechanics. There is a broad spectrum of coordinate grid types that can be created, either analytically or numerically; these grids can be orthogonal, nonorthogonal, conformal, or nonconformal.

The most common approach to numerically construct curvilinear coordinate grids in two-dimensional or axisymmetric systems is to solve a pair of elliptic partial differential equations, usually subject to Dirichlet conditions at the four boundaries, for a pair of functions $x(\xi, \eta)$, $y(\xi, \eta)$ which represent a discrete mapping between the physical (x, y) space and the boundary-fitted curvilinear (computational) (ξ, η) space. A variation on methods of this type that has proven extremely powerful in the study of free surface flow problems in fluid dynamics is the strong constraint orthogonal mapping technique of Ryskin and Leal [3] (henceforth referred to as R & L).

The strong constraint method is a one-step method of mapping a discrete set of points that are evenly distributed inside a unit square in the (ξ, η) curvilinear coordinate domain onto a discrete set of points in (x, y). The two coordinate grids are related via the metric tensor, and to ensure orthogonality of the generated grid in (x, y), the off-diagonal components of the metric tensor are required to be zero. The diagonal components of the metric tensor—referred to as the scale factors by R & L—are denoted by h_ξ and h_η. The ratio $h_\eta(\xi, \eta)/h_\xi(\xi, \eta)$ is called the distortion function and is denoted by $f(\xi, \eta)$. In the strong constraint method of R & L the distortion function is specified as a function of position in the (ξ, η) domain $[0, 1] \times [0, 1)$ as *input* to the method to provide control over the grid spacing. In contrast, $f(\xi, \eta) \equiv 1$ for a conformal mapping.

The most important limitation of the strong constraint method, in our opinion, is that no *a priori* criteria was given for choosing a distortion function $f(\xi, \eta)$ that

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guarantees the existence of an orthogonal map. In previous applications of the method, we have utilized functions of the product form

$$f(\xi, \eta) = \Phi(\xi)\, \Theta(\eta) \tag{1}$$

but this choice was *ad hoc* and the existence of an orthogonal map was only demonstrated, after the fact, by numerical construction. In the present note, we provide a simple proof that if f is of the form (1), then the mapping equations with their associated boundary conditions comprise a well-posed problem and the existence of an orthogonal coordinate grid is guaranteed (subject, of course, to discretization error in the numerical implementation and certain necessary restrictions on Φ and Θ). We do not claim that the use of a distortion function of product form is a necessary condition for the existence of an orthogonal map, only that it is sufficient.

Qualitatively, a distortion function of this form corresponds to a stretch of coordinate lines ξ independent of η, and a stretch of η lines independent of ξ. Thus, it is not only sufficient, but appears as a natural choice for construction of a boundary-fitted coordinate map which allows control of spacing between coordinate lines while still remaining orthogonal.

2. Formulation

In the strong constraint method, as formulated by R & L, the mapping functions $x(\xi, \eta)$, $y(\xi, \eta)$ relating the natural Cartesian coordinates (x, y) and orthogonal boundary-fitted coordinates (ξ, η) are obtained as the solution of a pair of covariant Laplace equations

$$\begin{aligned} \frac{\partial}{\partial \xi}\left(f \frac{\partial x}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{1}{f}\frac{\partial x}{\partial \eta}\right) &= 0 \\ \frac{\partial}{\partial \xi}\left(f \frac{\partial y}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{1}{f}\frac{\partial y}{\partial \eta}\right) &= 0. \end{aligned} \tag{2}$$

in which the distortion function $f(\xi, \eta)$ is specified in advance to provide control over the grid spacing. Following the conventions of R & L we consider the mapping problem on a bounded domain, with the coordinate line $\xi = 1$ corresponding to the free surface Γ^*, while $\xi = 0$ corresponds to the origin in the (x, y) space. If a mapping is required on an unbounded domain *exterior* to Γ^*, a preliminary conformal inversion can be applied to convert the exterior problem to an equivalent interior problem. It is therefore sufficient here to consider only the bounded domain, interior case. The coordinate ξ is clearly of a generalized radial type, while

η is the corresponding orthogonal, angular coordinate. We define η such that $0 \leqslant \eta < 1$, and require periodicity, that is,

$$(x(\xi, 1), y(\xi, 1)) = (x(\xi, 0), y(\xi, 0)), \qquad \xi \in [0, 1]. \tag{3a}$$

The origin is defined by

$$(x(0, \eta), y(0, \eta)) = (0, 0), \qquad \eta \in [0, 1), \tag{3b}$$

and we require, in addition,

$$y(1, 0) = 0. \tag{3c}$$

The conditions (3b) and (3c) fix a particular translation of the coordinates, and the "starting point" of the angular coordinate η, respectively.

In free-boundary problems, the geometry of the free surface is unknown and obtained as part of the solution. In the present framework, this boundary is denoted by

$$\Gamma^* = \{x(1, \eta), y(1, \eta) \colon \eta \in [0, 1)\}. \tag{4}$$

An obvious generic approach to determining $x(\xi, \eta)$, $y(\xi, \eta)$ is via an iterative process starting from some initial guess for the free surface shape. However, this cannot be done by simply incrementing $x(\xi, \eta)$ and $y(\xi, \eta)$ directly at each step, because this overdetermines the map when both $f(\xi, \eta)$ and orthogonality are specified. Instead, R & L developed a procedure for generating successive values of $(\partial x/\partial \xi)_{\xi=1}$ and $(\partial y/\partial \xi)_{\xi=1}$ at each step, by incrementing the scale factor $h_\xi(1, \eta)$. The scale factors of the system are defined by

$$h_\xi \equiv \left[\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2\right]^{1/2}$$

$$h_\eta \equiv \left[\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2\right]^{1/2}.$$

Hence, the problem is to obtain a pair of mapping functions x, y by solution of Eq. (2) subject to conditions (3) for specified values of the scale factor $h_\xi(1, \eta)$. The question is whether an orthogonal map exists for the particular choice of distortion function $f(\xi, \eta)$ given by Eq. (1).

3. The Proof

It must be shown that there exists a transformation path between the unit square in the (ξ, η) domain and the desired domain $\mathscr{D}$ in the (x, y) plane. This will be

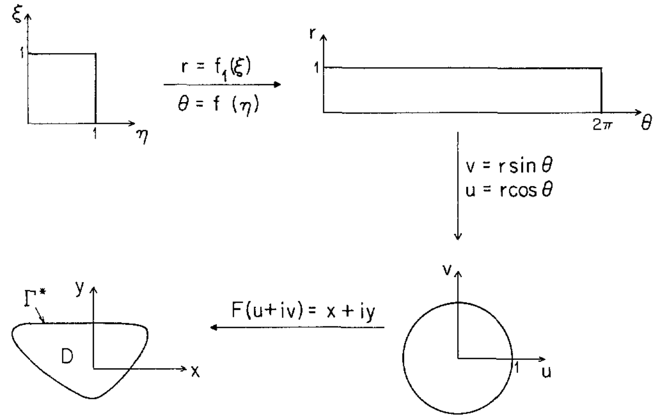

FIG. 1. Schematic representation of the transformation path between the finite domain and a unit square in the computational domain.

accomplished by breaking the path up into three steps (see Fig. 1), with the first step being a change of variables $r=f_1(\xi)$ and $\theta=f_2(\eta)$, from a unit square in the (ξ, η) computational domain to a rectangle in a circular cylindrical-type (r, θ) coordinate domain (maintaining the radial sense of ξ and the angular sense of η). The functions $f_1(\xi)$ and $f_2(\eta)$ are assumed to be one-to-one and smooth,[1] and are normalized and shifted for convenience so that $f_1(0)=0$, $f_1(1)=1$, $f_2(0)=0$, and $f_2(1)=2\pi$. Beyond these elementary restrictions f_1 and f_2 are arbitrary. The function f_1 represents a stretching or shrinking of the ξ coordinate lines, independent of η, and likewise, f_2 causes a stretching or shrinking of η coordinate lines independent of ξ. The functions f_1 and f_2 will be used to construct the distortion function f. In this context, freedom in choosing f_1 and f_2 corresponds to control over grid spacing. As mentioned above, the coordinate variables r and θ are circular cylindrical-type, and so the rectangle in (r, θ) can be transformed to a unit disk in (u, v) space using the relations $u=r\cos\theta$ and $v=r\sin\theta$. The motivation for carrying out these two coordinate transformations (from (ξ, η) to (r, θ) to (u, v)) arises from complex variable theory, where the Riemann mapping theorem guarantees the existence of a conformal map connecting a *given*, nontrivial two-dimensional domain to a unit disk. We have an analogous situation here except that for the strong constraint method the actual domain $\mathscr{D}$ in (x, y) is *not* known, since the boundary Γ^* of $\mathscr{D}$ is itself unknown. What is known instead is $h_\xi(1, \eta)$.

Consider a particular choice for f, namely

$$f(\xi, \eta)=\frac{f_1(\xi)\, f_2'(\eta)}{f_1'(\xi)}, \tag{5}$$

[1] In theory, f_1 and f_2 should be C^∞ but in numerical implementation this restriction may be relaxed.

where $f'_1(\xi)$ and $f'_2(\eta)$ refer to differentiation with respect to ξ and η, respectively. For this choice of distortion function, Eq. (2) on the unit disk reduces to

$$\frac{\partial^2 x}{\partial u^2}+\frac{\partial^2 x}{\partial v^2}=0$$
$$\frac{\partial^2 y}{\partial u^2}+\frac{\partial^2 y}{\partial v^2}=0. \tag{6}$$

The distortion function $f(\xi, \eta)$ is a product of the functions $f_1(\xi)/f'_1(\xi)$ and $f'_2(\eta)$, and the form may appear confusing at first glance due to the term f_1 in the numerator. However, this term is present because it is the necessary "length" factor required when dealing with angular and radial type coordinate systems.

The form of Eq. (6) suggests that we seek an analytic function F of the form $F(w) \equiv F(u+\imath v) = x + \imath y$ on the closed unit disk $u^2+v^2 \leqslant 1$. This function maps the unit disk onto the domain $\mathscr{D}$. Direct manipulation shows that specification of $h_\xi(1, \eta)$ is equivalent to specifying $|F'|$, the norm of the derivative of the analytic function, on the boundary of the disk. Note that $h_\xi(1, \eta)$ is assumed to be neither zero nor unbounded for the problem as posed to make sense. As a result of the Cauchy–Riemann equations

$$\frac{\partial x}{\partial u}=\frac{\partial y}{\partial v}, \qquad \frac{\partial x}{\partial v}=-\frac{\partial y}{\partial u},$$

Eq. (6) is identically satisfied. Further, Eqs. (3a)–(3c) reduce to the very simple constraints

$$F(0)=0 \tag{7a}$$

and

$$\arg\{F(1)\}=0. \tag{7b}$$

The analytic function $F(w)$ is required to be invertible, so in addition to the above three constraints we impose $|F'| \neq 0$ on the closed disk.

As mentioned earlier, if Γ^* (or equivalently, the value of F on the boundary of the disk) is known, then the Riemann mapping theorem guarantees the existence of the conformal mapping $F(w)$, and therefore the existence of an orthogonal mapping between (ξ, η) and (x, y). Here instead we have $|F'|$ specified on the boundary, and it is necessary to prove the existence of $F(w)$. This is carried out as follows: first, define the function

$$G(u+\imath v)=\operatorname{Log}(F')=\ln|F'|+\imath \arg\{F'\},$$

where the $k=0$ branch of the log has been selected. The function F is assumed analytic on the disk, implying that F', and therefore G,[2] is also analytic on the disk. As a result, specification of $|F'|$ on the boundary (of the disk) is equivalent to specification of $\Re\{G\}$ on the boundary. Poisson's formula immediately gives $\Re\{G\}$ in the interior of the disk ($r<1$):

$$\Re\{G(re^{\imath\theta})\} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-\alpha)+r^2}\,\Re\{G(e^{\imath\alpha})\}\,d\alpha. \tag{8}$$

Then, $\Im\{G\} = \arg\{F'\}$ is determined in the disk via the Cauchy–Riemann equations for G,

$$\frac{\partial\Im\{G\}}{\partial v} = \frac{\partial\Re\{G\}}{\partial u}, \qquad \frac{\partial\Im\{G\}}{\partial u} = -\frac{\partial\Re\{G\}}{\partial v}, \tag{9}$$

and subsequent direct integration. Formally, $\arg\{F'\} = s(u, v) + c_1 = \mathscr{S}(u+\imath v) + c_1$, where c_1 is a real constant resulting from the integration. The total derivative $F'(w)$ is then recovered as $F' = |F'|\exp[\imath(\mathscr{S} + c_1)]$. Analyticity of F' implies the existence of a Taylor series representation in the disk

$$F'(w) = e^{\imath c_1}(a_0 + a_1 w + \cdots), \tag{10}$$

where $w = u + \imath v$. A subsequent complex integration of $F'(w)$ yields $F(w)$ to within a second complex constant, c_2:

$$F(w) = c_2 + e^{\imath c_1}(a_0 + \tfrac{1}{2}a_1 w + \cdots)w. \tag{11}$$

Application of condition (7a) yields $c_2 = 0$, and the real constant c_1 represents the rotational orientation of the mapping and is fixed by condition (7b). Therefore, the mapping $F(w)$ does exist subject to the constraints mentioned above. Thus, by a route involving two coordinate transformations and one conformal map, we have shown that an orthogonal mapping between (ξ, η) and (x, y) *does* exist, and that it is determined by specification of $h_\xi(1, \eta)$ along with a special product form for the distortion function f.

4. Conclusions

One very important consideration in the strong constraint method of R & L is the relationship between the choice of f and the existence of an orthogonal mapping. It is intuitive that an arbitrary stretching of a conformal map will yield a nonorthogonal mesh, or mapping, and further that a solution to Eq. (2) may not

[2] Since $|F'| \neq 0$ on the disk, G is well defined.

even exist for certain choices of f. In this note we have shown that if f is of a special product form, represented by Eq. (1) and if h_ξ is specified at one boundary, then an orthogonal mapping *does* exist between (ξ, η) and (x, y).

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